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Some comments on recent generalizations of Cattaneo–Mindlin

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Recently, Dr. Ciavarella published three articles (Ciavarella, 1998a,b, 1999), in which the subject is a theory for frictional contact of equal half-planes and axisymmetric half-spaces, which was previously published by Jäger (1995, 1997a,b). Jäger's solution also covers the case of proportional and incremental loading scenarios. Dr. Ciavarella received these publications in April 1998, but he did not relate his results to Jäger's theory.

In the present note, it is shown that Ciavarella's calculation of the tangential tractions, forces and displacements is not necessary. Further, the example of wedge and conical indenters can be written as a superposition of a Hertzian and a flat profile with rounded corners. The solution for flat rounded profiles was published by Schubert (1942). In the first section of this note, Jäger's theory is shortly summarized. Some comments on Ciavarella's theory for axisymmetric contact follow in Section 2 and the last section treats Ciavarella's theory for plane contact. In order to keep the present note short, we discuss only the surface traction and displacements for a single contact area. Ciavarella's examples for multiple plane contact and the sub-surface stress for axisymmetric contact have not been examined. More remarks can be found in another discussion by Jäger (1999c).

1. Jägers theory for elastic friction

Historically, the first generalization of Cattaneo–Mindlin was published by Jäger (1995) for axisymmetric contact. From that publication follows the generalization for plane elasticity with multiple contact areas, which was presented at the GAMM-meeting in April 1996 (Jäger, 1997a). It was also shown by Jäger (1999a,b) that this theory can be used for thin and thick layers and some conditions for the generalization of Cattaneo–Mindlin have been presented at the GAMM meeting (Jäger, 1999d).

In order to simplify Ciavarella's solution, we will summarize the theory by Jäger shortly. For two bodies in contact, the combined displacements are the sum of the displacements of each body. The contact condition for a normal pressure increment is a constant combined displacement of the initial contact area. The

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stick condition in the tangential direction is also a constant rigid body displacement in the stick area. In the slip area, Coulomb's law requires the tangential traction to be proportional to the normal pressure. When the normal and tangential stress–displacement equations are identical, with the exception of a constant factor, each normal solution is also a tangential solution. In this case, the tangential traction q is the difference of the full slip stress of the contact area fp and the slip stress of the stick area fp^*, with Coulomb's coefficient of friction f. Variables with asterisk characterize values of the stick area; and the stick areas for increasing tangential forces Q are identical with the contact areas for decreasing normal forces P. The force is an integral of the traction. The tangential displacement α_t is also proportional to the normal displacement α_n of the contact area and α_n^* of the stick area, with an appropriate stiffness factor κ

$$q = f(p - p^*), \qquad Q = f(P - P^*), \qquad \alpha_t = f\kappa(\alpha_n - \alpha_n^*). \tag{1}$$

In the case of an axisymmetric contact area with the radius a, all the values with an asterisk can be written as a function of the radius of the stick area a^*, i.e. $p^* = p(a^*)$. The relation $q^* = fp^*$ is missing in the mentioned publications by Ciavarella. In the next section, we discuss Ciavarella's publication on axisymmetric contact.

2. Axisymmetric contact

Ciavarella (1999) summarized his theory in Eqs. (36) for the corrective shear force and Eq. (38) for the tangential displacement, which are identical with Eqs. (33) and (34) by Jäger (1995). For the special case of a flat punch with rounded corners, the gap $z_1(r)$ between the surfaces in undeformed contact has the form

$$z_1(r) = \begin{cases} 0, & 0 \leqslant r \leqslant b, \\ \frac{1}{2R_c}(r-b)^2, & b \leqslant r \leqslant a, \end{cases} \tag{2}$$

where R_c, is the rounding radius, b, the radius of the flat area and a, the contact radius. Ciavarella obtained the solution for the normal force $P_1(a)$, the normal displacement $\alpha_{n1}(a)$ and the pressure $p_1(a)$. We shortly summarize these formulas in a modified form, because they are useful below:

$$3AR_cP_1(a) = 2(4a^2 - b^2)\sqrt{a^2 - b^2} - 6ba^2 \arccos\frac{b}{a}, \tag{3}$$

$$R_c\alpha_{n1}(a) = a\sqrt{a^2 - b^2} + ab \arccos\frac{b}{a}, \tag{4}$$

$$\pi AR_c p_1(a,r) = \int_h^a \left(2\sqrt{s^2 - b^2} - b \arccos\frac{b}{s}\right)\frac{2\,\mathrm{d}s}{\sqrt{s^2 - r^2}}, \quad h = \begin{cases} b, & 0 \leqslant r \leqslant b, \\ r, & b \leqslant r \leqslant a, \end{cases} \tag{5}$$

$$A = \frac{1 - \nu_I}{G_I} + \frac{1 - \nu_{II}}{G_{II}}, \tag{6}$$

where the variables ν_k and G_k denote Poisson's ratio and the shear modulus of body k, respectively. It seems that the integral with the term $\arccos(b/s)$ in Eq. (5) cannot be expressed with tabulated functions, but it is possible to evaluate the derivative $\mathrm{d}p(r)/\mathrm{d}r$. Using integration by parts, the square-root singularity under the integral must be eliminated first. The resulting integral can easily be differentiated which gives some elliptic integrals listed in Gradstein and Ryshik (1981)

$$\frac{1}{2}\pi Ar\frac{\mathrm{d}p(a,r)}{\mathrm{d}r} = \frac{a}{\sqrt{a^2 - r^2}}\left(b \arccos\frac{b}{a} - 2\sqrt{a^2 - b^2}\right) + 2h[F(\nu,t) - E(\nu,t)] - \frac{b^2}{h}F(\mu,t) + 2a\sin\mu, \tag{7}$$

	h	t	μ	v
$r \leqslant b$	b	r/b	$\arcsin\sqrt{(a^2-b^2)/a^2-r^2)}$	$\arcsin b/a$
$r \geqslant b$	r	b/r	$\arcsin\sqrt{(a^2-r^2)/(a^2-b^2)}$	$\arcsin r/a$

$F(v,t)$ and $E(v,t)$ are elliptic integrals of the first and second kind. Eq. (7) has the remarkable property that the slope of the pressure is infinite at the border of the stick area ($r=b$, $t=1$, $\mu=\pi/2$) and the contact area ($r=a$). The first singularity is logarithmic, and is not directly visible in Figs. 2 and 3, which Ciavarella (1999) obtained by numerical integration. Integral (5) was also published by Schubert (1942), who presented an approximation for small roundings ($a \to b$), which is identical with the corresponding solution for plane contact.

In Section 3.2, Ciavarella discussed the example of a conical punch with rounded tip

$$z_2(r) = \begin{cases} \frac{r^2}{2R_c}, & 0 \leqslant r \leqslant b, \\ \frac{b}{2R_c}(2r-b), & b \leqslant r \leqslant a, \end{cases} \tag{8}$$

Profile (8) can be written as the difference of a Hertzian profile and a flat punch (2):

$$z_2(r) = \frac{r^2}{2R_c} - z_1(r). \tag{9}$$

In linear elasticity, solutions can linearly be superposed, and it is not necessary to perform any calculation for the solution of this case. The Hertzian result is the special case $b=0$ in Eqs. (3)–(5). Thus, we obtain the result

$$P_2(a) = \frac{8a^3}{3AR_c} - H\left(\frac{a}{b}\right)P_1(a), \tag{10}$$

$$\alpha_{n2}(a) = \frac{a^2}{R_c} - H\left(\frac{a}{b}\right)\alpha_{n1}(a), \tag{11}$$

$$p_2(a,r) = \frac{4\sqrt{a^2-r^2}}{\pi AR_c} - H\left(\frac{a}{b}\right)p_1(a,r), \tag{12}$$

$$H(x) = \begin{cases} 0, & x<1, \\ 1, & x \geqslant 1, \end{cases} \tag{13}$$

where the index 2 characterizes values for the punch with rounded tip and index 1 values from Eqs. (3)–(6). Eqs. (10)–(13) are identical with Ciavarella's Eqs. (19)–(21). For $a \leqslant b$, the contact area a is smaller than the rounded part b of the tip, and the contact is Hertzian.

In Section 4.1, Ciavarella (1999) discussed the tangential solution for a flat rounded punch. Again, there is no necessity for calculation and the results are given by Eq. (1). For completeness, we summarize the results shortly

$$q_1(a^*,a,r) = fp_1(a,r) - fp_1(a^*,r), \tag{14}$$

$$Q_1(a^*,a) = fP_1(a) - fP_1(a^*), \tag{15}$$

$$\alpha_{t1} = f\kappa[\alpha_{1n}(a) - \alpha_{1n}(a^*)], \tag{16}$$

$$\kappa A = \frac{2-\nu_{\rm I}}{2G_{\rm I}} + \frac{2-\nu_{\rm II}}{2\,G_{\rm II}}. \tag{17}$$

Eq. (17) holds only for axisymmetric contact. Formulas (14)–(17) can also be used for the conical punch with rounded tip, after substitution of index 2 for index 1. Eqs. (14)–(17) cover all cases of Eqs. (39)–(47) by Ciavarella (1999).

3. Plane contact

The theory for plane contact with multiple contact areas was reported by Jäger (1997a) and proved in Jäger (1998). Ciavarella (1998a) summarized his theory in Eqs. (18) and (19), which are better expressed with Eq. (9) $q^* = fp^*$ by Jäger (1997a). This relation, and the resulting tangential force $Q^* = fP^*$, are missing in Ciavarella (1998a,b). Some examples by Ciavarella (1998b) have also been published by Jäger (1997b), and a comparison of identical formulas is given in the table below:

Ciavarella (1998b)	(3)	(9)	(A9)	(A10)
Jäger (1997b)	(3)	(40)	(18)	(17)

Eqs. (22) and (23) by Ciavarella (1998b), for the flat punch with rounded corners, have been written in the form by Schubert (1942)

$$\pi A R_{\rm c} p_1(a,x) = 2\sqrt{a^2-x^2}\arccos\frac{b}{a} + (x-b)\ln\left|\frac{u-xb}{a(x-b)}\right| - (x+b)\ln\left|\frac{u+xb}{a(x+b)}\right|, \tag{18}$$

$$u = a^2 + \sqrt{(a^2-b^2)(a^2-x^2)}, \tag{19}$$

$$A R_{\rm c} P_1(a) = a^2\arccos\frac{b}{a} - b\sqrt{a^2-b^2}, \tag{20}$$

$$A = \begin{cases} \frac{1-\nu_1}{G_1} + \frac{1-\nu_2}{G_2}, & \text{plane strain,} \\ \frac{2}{E_1} + \frac{2}{E_2}, & \text{plane stress.} \end{cases} \tag{21}$$

As in the case of axisymmetric contact, the slope of the pressure is infinite at $r = b$. The solution for a wedge with rounded apex is the difference of a Hertzian profile and flat punch (18)–(20)

$$P_2(a) = \frac{\pi a^2}{2AR_{\rm c}} - H\left(\frac{a}{b}\right)P_1(a), \tag{22}$$

$$p_2(a,x) = \frac{\sqrt{a^2-x^2}}{AR_{\rm c}} - H\left(\frac{a}{b}\right)p_1(a,x) \tag{23}$$

with $H(x)$ given by Eqs. (13) and $P_1(a)$, $p_1(a,x)$ by Eqs (18)–(20). Again, the tangential solution for symmetric profiles follows from Eqs. (14)–(17), with $\kappa = 1$ for plane contact. The displacements can only be determined relative to a reference point, as usual for half-planes.

It should be noted that Ciavarella's Eq. (A9) has the wrong dimension and becomes singular for odd k. It should be replaced with Eqs (18) by Jäger (1997b)

$$p(a,x) = \frac{kP(a)}{\pi a^2}\sqrt{a^2 - x^2}F\left(1, \frac{2-k}{2}; \frac{3}{2}; 1 - \frac{x^2}{a^2}\right) \quad \text{for } z = |x|^k \tag{24}$$

with the hypergeometric function $F(a, b; c; x)$.

Ciavarella (1998a) distinguishes among known or unknown contact areas on page 2355, but only the profile is known, a priori. When the profile has edges with vertical or concave tangents, such edges are a border of the contact area. In this case, a rigid body displacement in form of a flat punch solution with sharp edges can be superposed. When the slope of the profile is finite and convex, the pressure falls to zero at the border of the contact area.

The remarks on page 2356 by Ciavarella (1998a) should be replaced with the following:

- The corrective shear traction q^* is always the full slip stress fp^* of the stick area.
- The stick condition is a constant rigid body displacement, and therefore, identical only with the contact condition, when the bodies do not rotate. Consequently, the bodies must not rotate during tangential loading.
- For contact profiles with flat areas of different height, higher areas come into contact with increasing normal forces and the new contact areas change their form with the compression. When a tangential force is applied, a slip area propagates at the highest flat area; a partial slip solution exists. Only when all flat areas have the same height, all contact areas stick completely for forces below Coulomb's limit.

Finally, it can be concluded, from the pressure (7) and (18) for flat rounded punches, that a discontinuous second derivative of the displacement produces an infinite slope of the pressure.

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